Exercises in functional iteration: the function $f(x) = \ln(2-\exp(-x))$

A selfstudy using formal powerseries and operator-matrices
Gottfried Helms 10.12.2010
update 12.02.2011

1. Definition

The function considered is an example taken from a private conversation with D.Geisler and W.Jagy. Here I discuss properties of fractional iteration of

$$f(x) = \ln(2-\exp(-x))$$
$$f^{1}(x) = f(x)$$
$$f^{0}(x) = x$$
$$f^{h+1}(x) = f^{h}(f(x)) \quad // \text{meaning the } h\text{'th iterate}$$
$$f^{-1}(x) = -\ln(2-\exp(x)) = -f(-x) \quad // \text{mirroring around the origin}$$

A rough idea can be got by the following plot. The function has real range only for $-\ln(2)<x$; this is the right red curve in the following. For $x\to\infty$ it approximates the constant $\ln(2)$. For $x<\ln(2)$ the function has the imaginary component $\pi i$, (green lines) and its real part approaches the function $f(x)=-x$ for $x\to-\infty$ (left red curve)

Because there is the fixpoint $f(0)=0$ we can construct formal powerseries for arbitrary continuous iterates and because $f'(x)<1$ for $x>0$ it is an attracting fixpoint and iteration to positive heights is a very well converging process. Here the coefficients $c_{k,h}$ for the formal powerseries of a certain iteration-height $h$ can be taken by a set of polynomials in $h$ which shall be determined later:

$$f^{h}(x) = c_{1,h}x + c_{2,h}x^{2} + c_{3,h}x^{3} + \ldots$$

We shall find, that each coefficient $c$ is a polynomial in the $h$-parameter which can be explicitely be determined (without need of recursion)
2. **Procedere:**

2.1. **Construct the matrix-operator for the function**

\[ f(x) = \ln(2 - \exp(-x)) \]

The powerseries for this can be given in rational coefficients:

\[ f(x) = x - x^2 + \frac{13}{12}x^3 + \frac{5}{4}x^4 - \frac{541}{360}x^5 + \frac{223}{120}x^6 + O(x^8) \]

The associated matrix-operator \( M \) (of infinite size) begins with

\[
M = \begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & \cdot & \cdot & \cdot \\
0 & -1 & 1 & \cdot & \cdot \\
0 & 1 & -2 & 1 & \cdot \\
0 & -\frac{13}{12} & 3 & -3 & 1 \\
0 & \frac{5}{4} & -\frac{25}{6} & 6 & -4 & 1 \\
0 & -\frac{541}{360} & 17/3 & -\frac{41}{4} & 10 & -5 & 1 \\
0 & \frac{223}{120} & -\frac{1381}{180} & 65/4 & -61/3 & 15 & -6 & 1 \\
\end{bmatrix}
\]

We see, that the coefficients for \( f(x)^0 \) are in the first column (columnindex \( c=0 \)), that for \( f(x)^1 \) in the second column (\( c=1 \)), that for \( f(x)^2 \) in the third column (\( c=2 \)) and so on.

2.2. **The use of the matrix-operator for expression of the required powerseries**

With a "Vandermonde"-like vector-type

\[ V(x) = \text{row}( 1, x, x^2, x^3, ...) \]

we can write

\[ V(x) \ast M = V(f(x)) = \text{rowvector}(1, f(x), f(x)^2, f(x)^3, ...) \]

which immediately allows generalization for iterates of \( f(x) \).

Let’s write the \( h \)’th iterate \( f^h(x) \). Then we have, first for integer heights:

\[ V(x) \ast M^h = V(f^h(x)) \]

where integer powers of \( M \) can exactly be determined by matrix-multiplication of the triangular matrix with itself. This allows still exact rational arithmetic up to arbitrary truncation size.

The question is now: can also fractional iterates be determined. The answer is yes; and this is a known procedure.
2.3. fractional powers of a matrix-operator via log/exp

Since $f(x)$ is a function having $f(0) = 0, f'(0) = 1$ we can determine a matrix-logarithm which still provides exact arithmetic:

\[
\begin{align*}
\text{define } M1 &= M - \text{ID} & (\text{where ID is the identity-matrix}) \\
\text{Log}(M) &= M1 - M1^2/2 + M1^3/3 - ... + ... & (\text{the mercator-series for log})
\end{align*}
\]

Then

\[
\begin{align*}
L &= \text{Log}(M) \\
M^h &= \text{Exp}(h \times L)
\end{align*}
\]

provides the $h$th-power of $M$, again in rational arithmetic.

The logarithm-matrix $L$ has an interesting format:

\[
L = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & -1/12 & 0 & -3 & 0 & 0 \\
0 & 0 & -1/5 & 0 & -4 & 0 \\
0 & -1/360 & 0 & -1/4 & 0 & -5 \\
0 & 0 & -1/180 & 0 & -1/3 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The logarithm of an operator-matrix has always the structure that all columns are simply shifted multiples of the second column; we see, that the coefficients $L_{rc}$ in some col $c$ and row $r$ are just $cL_{r+1,c,1}$. (Thus the matrix $L$ is not an operator-matrix!)

And the second column ($c=1$) gives the coefficients for a function, which we may call the "iterative-logarithm-function of $f(x)$", as a function we get:

\[
lf(x) = -2(x^2/2! + x^4/4! + x^6/6! + ...)
\]

which is also

\[
lf(x) = -2(\cosh(x) - 1) = 2 - (\exp(x) + \exp(-x))
\]

A factorially scaled version of matrix $L$ is

\[
dF \times L \times dF^{-1} = FLf
\]

and begins with

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -6 & 0 & 0 & 0 \\
0 & -2 & 0 & -12 & 0 & 0 \\
0 & 0 & -10 & 0 & -20 & 0 \\
0 & -2 & 0 & -30 & 0 & -30 \\
0 & 0 & -14 & 0 & -70 & 0 & -42 \\
\end{bmatrix}
\]

If we want to compute some general power $h$ of the matrix $M$ we have to evaluate the exponential:

\[\text{I've seen the term "schlicht"-function for this in older literature (german for "simple function")}\]
Iteration of \( f(x) = \ln(2 - \exp(-x)) \)

2.4. The half-power of \( M \) (using \( h=\frac{1}{2} \)) give the half-iterate of \( f(x) \)

We can do this for some actual value of \( h \), say \( h=\frac{1}{2} \) to get the powerseries for the half-iterate. The top left segment of the matrix is

\[
M^{0.5} = \exp(\frac{1}{2} \log(M))
\]

and in the second column we find the coefficients of the formal powerseries for \( f^{0.5}(x) \):

\[
f^{0.5}(x) = x - 1/2*x^2 + 1/4*x^3 - 1/6*x^4 + 1/8*x^5 - 137/1440*x^6 + 71/960*x^7 + O(x^8)
\]

such that

\[
V(x) \cdot M^{0.5} = V(f^{0.5}(x))
\]

or, with a bracket-[row,col]-notation for the extraction of the second column of matrix \( M^{0.5} \) we write

\[
f^{0.5}(x) = V(x) \cdot M^{0.5}[1]
\]

2.5. The general power of \( M \) and the general iterate of \( f(x) \); symbolically

If we want this in more generality, keeping the iteration-height parameter \( h \) as variable we can compute the matrix-exponential symbolically getting the following polynomials in \( h \) as entries of the \( h \)'th power \( M^h \):

\[
M^h = \exp(h \cdot \log(M))
\]

where the second column \( M^h[1] \) only is needed to provide the relevant polynomials for the computation of \( f^h(x) \).

If we insert, for instance, \( h=1 \), we get the original powerseries for \( f(x) \), if we insert \( h=\frac{1}{2} \) we get the coefficients of the powerseries for the half-iterate \( f^{0.5}(x) \) and so forth.
2.6. The bivariate coefficients-matrix \textbf{POLY}

If the coefficients of \( h \) in that second column are again represented as a matrix, we can write this as matrix \textbf{POLY} of coefficients for the bivariate function \( f^h(x) = V(x) \cdot \textbf{POLY} \cdot V(h)\sim \)

\[
\textbf{POLY} = \begin{bmatrix}
0 & \cdot & \cdot \\
1 & 0 & \cdot \\
0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1/12 & 0 & -1 & 0 \\
0 & 0 & 1/4 & 0 & 1 & 0 \\
0 & -1/360 & 0 & -1/2 & 0 & -1 & 0 \\
0 & 0 & 1/40 & 0 & 5/6 & 0 & 1
\end{bmatrix}
\]

So \textbf{POLY} \cdot V(h)\sim gives the second column in \( M^h \) and as a bivariate expression in the matrix-notation we have:

\[
f^h(x) = V(x) \cdot \textbf{POLY} \cdot V(h)\sim \\
= x \cdot \left(1\right) \\
+ x^2 \cdot \left(-1h\right) \\
+ x^3 \cdot \left(1h^2\right) \\
+ x^4 \cdot \left(-12h - 1h^3\right) \\
+ x^5 \cdot \left(1/4h^2 + 1h^4\right) \\
+ x^6 \cdot \left(\ldots\right) \\
+ \ldots
\]
2.7. Explicite descriptions of entries in \textit{POLY}

It might be of interest, that \textit{POLY} can be rescaled to provide integer entries only (heuristically, no proof yet). This is possible using a factorial (similarity) scaling; here is the top-left segment (assume \(dF\) as diagonal matrix of factorials \(\text{diag}([0!,1!,2!,...])\))

\[
dF \ast \text{POLY} \ast dF^{-1} = \text{FPf} =
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 \\
0 & -2 & 0 & -4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 15 & 0 \\
0 & 0 & 0 & 60 \\
0 & 0 & 63 & 0 \\
0 & 1 & 75 & 0 \\
0 & 0 & 7 & 0
\end{bmatrix}
\]

I succeeded in finding a general expression for the entries in \(\text{FPf}\) and thus for that in \textit{POLY}. Assuming the matrix-indices \(r(\text{ow})\) and \(c(\text{ol})\) beginning at zero we have for the elements in \textit{POLY}:

\[p_{r,c} = \frac{1 - (-1)^{r-c}}{r!} \sum_{k=1}^{c} \left( -1 \right)^k \binom{c-1}{k} \cdot k^{r-1}\]

or with the binomial-coefficient more conveniently adapted:

\[p_{r,c} = \begin{cases} 
\delta_{r,1} & \text{if } c = 0 \\
\frac{1 - (-1)^{r-c}}{r!} \cdot \frac{1}{c} \sum_{k=1}^{c} (-1)^k \binom{c}{k} \cdot k^{r-1} & \text{if } c > 0
\end{cases}
\]

\textit{where d is the Kronecker-symbol}

We see that the \(p_{r,c}\) are finitely composed polynomials whose number of terms is just equal to the column-index \(c\). So to describe the powerseries for the \(h\)'th iterate of \(f(x)\) we write:

\[f^h(x) = x + (p_{2,1} \ast h) x^2 + (p_{3,2} \ast h^2) x^3 + (p_{4,1} \ast h + p_{4,3} \ast h^3) x^4 + ...\]

and because we do no more need the matrix-logarithm/matrix-exponential we can determine that coefficients to arbitrarily many terms in exact rational arithmetic and thus the whole function with optimal precision.
2.8. The function $w_x(h)$ (with fixed $x$) as powerseries in $h$

It might be of interest to reformulate the function to a fixed parameter $x$ where only the iteration $h$ is a variable argument. This means that we must introduce a family of functions $g(x,c)$ which depend on $x$ and use the coefficients of one column $c$ for its taylorseries. We shall then evaluate the functions $g(x,c)$ first giving the coefficients for the powerseries of $w_x(h)$ in terms of $h$ (which is actually only a rewriting as $w_x(h) = f^{|h|}(x)$) where

$$w_x(h) = \sum_{c=0}^{\infty} g(x, c) \cdot h^c$$

The functions $g(x,c)$ are first powerseries in $x$ using the entries $p_{r,c}$ along the columns of POLY:

$$g(x, c) = \sum_{r=1}^{\infty} x^r \cdot p_{r,c}$$

First we can change order of summation because each $p_{r,c}$ is a finitely composed sum of $c$ terms:

$$g(x, c) = \sum_{r=0}^{\infty} x^r \cdot \frac{1}{r!} \cdot \left(1 - (-1)^{r-c}\right) \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot k^r\right)$$

$$= \sum_{r=0}^{\infty} \frac{1}{c} \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot \left(1 - (-1)^{r-c}\right) \cdot \left(\frac{1}{r!}\right)\right)$$

$$= \frac{1}{c} \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot \sum_{r=1}^{\infty} \left(1 - (-1)^{r-c}\right) \cdot \left(\frac{k^r}{r!}\right)\right)$$

The inner sums can be reformulated into closed forms as exponentials:

$$g(x, c) = \frac{1}{c} \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot \left(e^{kx} - 1 - (-1)^c (e^{-kx} - 1)\right)\right)$$

This gives at even indexed columns $c=2j > 0$:

$$g_E(x, c) = \frac{1}{c} \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot \left(e^{kx} - 1 - (-1)^c (e^{-kx} - 1)\right)\right)$$

$$= \frac{2}{c} \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot \sinh(kx)\right)$$

and at odd indexed columns $c=2j+1$:

$$g_O(x, c) = \frac{1}{c} \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot \left(e^{kx} - 1 + (e^{-kx} - 1)\right)\right)$$

$$= \frac{2}{c} \cdot \sum_{k=1}^{c} (-1)^k \cdot \left(\binom{c}{k} \cdot \cosh(kx) - 1\right)$$
Expressed as function in Pari/GP this is

\[
\text{user-function } g(x,c) = \text{POLY-column_sum}
\]
\[
g(x,c) = \text{if}(c==0,\text{return}(x)); \text{if}(c \% 2, g_{\text{odd}}(x,c), g_{\text{even}}(x,c))
\]
\[
\text{\textbackslash internal functions}
\]
\[
g_{\text{even}}(x,c) = \frac{2}{c} \ast \text{sum}(k=1,c,(-1)^k \ast \binom{c}{k} \ast \sinh(k \ast x))
\]
\[
g_{\text{odd}}(x,c) = \frac{2}{c} \ast \text{sum}(k=1,c,(-1)^k \ast \binom{c}{k} \ast (\cosh(k \ast x) - 1))
\]

With this, the function \( w_h(x) = f^{\circ h}(x) \) is

\[
w_h(x) = \sum_{c=0}^{\infty} g(x,c) \ast h^c
\]

2.9. Numerical examples

At \( x=1 \) this gives numerically:

\[
w_1(h) = 1 - 1.08616126963 \ast h + 1.27645802056 \ast h^2 - 1.60687788465 \ast h^3
\]
\[
+ 2.13938758000 \ast h^4 - 2.97554022629 \ast h^5 + 4.27895075604 \ast h^6
\]
\[
- 6.31199338319 \ast h^7 + 9.49553583745 \ast h^8 + O(h^9)
\]

The coefficients seem to increase, but alternate in sign. Before further analysis if I apply Euler-summation I find results for fractional heights in the unit-interval \( 0 \leq h \leq 1 \)

<table>
<thead>
<tr>
<th>height h</th>
<th>( w_h(x) = f^{\circ h}(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.000000000000</td>
</tr>
<tr>
<td>0.05</td>
<td>0.948694725422</td>
</tr>
<tr>
<td>0.10</td>
<td>0.902729488855</td>
</tr>
<tr>
<td>0.15</td>
<td>0.861269948104</td>
</tr>
<tr>
<td>0.20</td>
<td>0.823653583114</td>
</tr>
<tr>
<td>0.25</td>
<td>0.789346386684</td>
</tr>
<tr>
<td>0.30</td>
<td>0.757912297060</td>
</tr>
<tr>
<td>0.35</td>
<td>0.728991136470</td>
</tr>
<tr>
<td>0.40</td>
<td>0.702282374504</td>
</tr>
<tr>
<td>0.45</td>
<td>0.677532970554</td>
</tr>
<tr>
<td>0.50</td>
<td>0.654528129922</td>
</tr>
<tr>
<td>0.55</td>
<td>0.633084178162</td>
</tr>
<tr>
<td>0.60</td>
<td>0.613042999976</td>
</tr>
<tr>
<td>0.65</td>
<td>0.594267650303</td>
</tr>
<tr>
<td>0.70</td>
<td>0.576638855091</td>
</tr>
<tr>
<td>0.75</td>
<td>0.560052195330</td>
</tr>
<tr>
<td>0.80</td>
<td>0.544415821523</td>
</tr>
<tr>
<td>0.85</td>
<td>0.529648584037</td>
</tr>
<tr>
<td>0.90</td>
<td>0.515678492527</td>
</tr>
<tr>
<td>0.95</td>
<td>0.50244137932</td>
</tr>
<tr>
<td>1.00</td>
<td>0.489880125645</td>
</tr>
</tbody>
</table>

Gottfried Helms, 10.12.2010