1. Fixpoint by construction (for "Tetration-Forum")

1.1. Intro

Denote tetration by the following recursive function:

\[(1.1.1)\]
\[T_s^{(0)}(x) = x \quad T_s^{(i)}(x) = s^{T_s^{(i-1)}(x)} \]

Note, that I begin to use the notation

\[T_s^{(oo)}(1) = s^{s^{s^{\ldots}}} \]

instead of the common

\[T_s^{(oo)}(1) = x^{x^{x^{\ldots}}} \]

because

* this seems more consistent with my matrix-approach to tetration and
* allows to define a starting value as top-exponent even for infinite towers (which seems also more consistent with the idea of an initial value for iterations (including infinite repetitions).
* this seems also to be more consistent with the notion of several fixpoints
* and with the problem of consistency of partial evaluation of an expression which is meant as being infinitely iterated.

Fixpoints

Since I don't have the Productlog-function ready in Pari/Gp, I used a handwaved recursive tracer to approximate complex "fixpoints" \( t \) for real bases \( s \) (and currently have to relate to this), such that

\[(1.1.2) \quad s^t = t \quad \text{or} \quad s = t^{1/t} \]

and

\[(1.1.3) \quad \ldots s^{s^{st}} = t \]

Here I want to have a deeper look into this problem. My approach is here, to assume an arbitrary complex \( t \) and see, whether we can construct all real \( s > 0 \) from this assumption. In fact, I actually assume a parameter \( u \), where \( u = \log(t) \) first, compute the unique \( t \) from this and then \( s \). I omit the periodicity for \( u \) for a start.

But the first problem is, can (1.1.2) actually be translated into (1.1.3), if \( t \) is complex?

1.2. Question: is \( \ldots x^x = y \) one-to-one translatable into \( y^x = x \) ?


\[(2) \quad [x^{(1/x)}]^{[x^{(1/x)}]^{[x^{(1/x)}]}} = \begin{cases} \text{x} & \text{if x is in } [1/e,e] \\ y, & \text{if x is in } (e,\infty) \end{cases} \]

where \( y \) is in \([1/e, e]\) and \( y \) satisfies: \( y^{(1/y)} = x^{(1/x)} \)

In a more graphical shape:

\[
a = x^x \Rightarrow \ldots a = \begin{cases} x & \text{if } -1 \leq \log(x) \leq 1 \\ y & \text{if } 1 < \log(x) \end{cases} \quad \text{where } -1 < \log(y) < 1 \quad \text{and } y^{1/y} = x^{1/x} = a
\]

In my understanding this first states the ambiguity of representations for \( a \). In the following the LambertW-function is discussed to establish the identity of (1.1.2) and (1.1.3). And then, working further through Ionannis' article, I got to the following consideration on my sketchpad.
1.3. Derivation

Assume the formal relation (in this example I use only the principal branch of logarithm)

\[(1.3.1.)\quad s = t^{1/t}\quad \text{and} \quad u = \log(t)\]

Assume the \(u\) as a free parameter and \(t\) and \(s\) depending on \(u\). Denote the components of \(t\) and \(u\)

\[(1.3.2.)\quad u = \alpha + \beta i\quad \text{and} \quad t = \exp(u) = a + b i\]

then first

\[(1.3.3.)\quad s = \exp\left(\frac{\alpha + \beta i}{a + bi}\right) = \exp\left(\frac{\alpha + \beta i}{a - bi}\right)\frac{1}{|t|^2}\]

Then to have \(s\) real, given the parameters \(u\) and \(t\) it is necessary that

\[(1.3.4.)\quad \text{either} \quad \beta = 0 \implies b = 0 \quad \text{(the "real-only-case")}\]
\[(1.3.5.)\quad \text{or} \quad \beta \neq 0 \quad \text{but} \quad (a\beta - b\alpha) = 0 \quad \text{(the "complex-to-real" case)}\]

By definition, \(a\) and \(b\) are functions of \(\alpha\) and \(\beta\), so we need only choose some \(\alpha\) and \(\beta\).

\[(1.3.6.)\quad t = a + bi = \exp(\alpha + \beta i) = \exp(\alpha) * (\cos(\beta) + i \sin(\beta))\]
\[(1.3.7.)\quad a = \exp(\alpha) * \cos(\beta) \quad b = \exp(\alpha) * \sin(\beta)\]

Then, to have \(s\) purely real it is required by \((1.3.3)\), that

\[(1.3.8.)\quad (a\beta - b\alpha) = 0 \implies a\beta = b\alpha\]
\[(1.3.9.)\quad \exp(\alpha) * \cos(\beta) - \exp(\alpha) * \sin(\beta)\alpha = 0\]

and since \(\beta\) is an argument of \(\cos(\cdot)\) and \(\sin(\cdot)\), it seems best to choose \(\beta\) as free parameter and \(\alpha\) as the dependent:

\[(1.3.10.)\quad \alpha = \beta \cos(\beta) / \sin(\beta)\]

so

\[(1.3.11.)\quad u = \beta \cos(\beta) / \sin(\beta) + \beta i = \beta / \sin(\beta) * (\cos(\beta) + \sin(\beta) i)\]

and a certain selection for \(\beta\) defines then the whole formula for \(s\).

We have then

\[(1.3.12.)\quad u = \frac{\beta}{\sin(\beta)} * \exp(\beta i) = \frac{\beta}{\sin(\beta)} * (\cos(\beta) + \sin(\beta) i)\]
\[(1.3.13.)\quad t = \exp(u)\]
\[(1.3.14.)\quad s = t^{\frac{1}{t}} = \exp\left(\frac{\alpha}{\exp(\alpha)}\right) = \exp\left(\frac{\alpha}{a} \right) = a^{\frac{j}{a}}\]

with singularities where \(\beta\) is a integer multiple of \(\pi\), with the one exception: the singularity at \(\beta=0*\pi\) is removed and \(\alpha\) can assume any value in this case. If \(\beta=0\) then \(\alpha\) is a free real parameter , \(u\) and \(t\) are then real, too, and we get the known form:

\[(1.3.15.)\quad s = t^{\frac{1}{t}} = \exp(\frac{u}{\tau}) = \exp\left(\frac{\alpha}{\exp(\alpha)}\right) = \exp\left(\frac{\alpha}{a} \right) = a^{\frac{j}{a}}\]
1.4. The "real-only" case (real $u,t,(\beta=0, b=0)$; real $s$)

If $\beta=0$ (then also $b=0$), then we may freely choose $\alpha$ having $u=\alpha$ and then $t = \exp(u) = \exp(\alpha)$ we have due to Euler, the special ranges for

\[
\begin{align*}
(1.4.1) & \quad s \text{ given by } 1/e^s < s < e^{1/e} \\
(1.4.2) & \quad 1/e < a=t < e \\
(1.4.3) & \quad -1 < \alpha = u < 1
\end{align*}
\]

Where $\alpha=1$ marks also the upper limit for $s$ in the above formula (1.3.15) (when setting $\beta=0$):

\[
(1.4.4) \quad s = t^{\frac{1}{t}} = \exp\left(\frac{\pi}{t}\right) = \exp\left(\frac{1}{\exp(1)}\right) = e^{\frac{1}{e}}
\]

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$t=a$</th>
<th>and $s$ evaluates to (limit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \to -\infty$</td>
<td>$a \to 0$</td>
<td>$s = e^{\frac{\alpha}{e}} = e^{\frac{0}{e}} \to e^{-\infty} = 0$</td>
</tr>
<tr>
<td>$\alpha = -1$</td>
<td>$a = 1/e$</td>
<td>$s = e^{\frac{-1}{e}} = e^{-e}$</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$a = 1$</td>
<td>$s = e^{\frac{0}{e}} = 1$</td>
</tr>
<tr>
<td>$\alpha = +1$</td>
<td>$a = e$</td>
<td>$s = e^{\frac{1}{e}} = e^{\frac{1}{e}}$ (maximum)</td>
</tr>
<tr>
<td>$\alpha \to +\infty$</td>
<td>$a \to \infty$</td>
<td>$s = e^{\frac{\alpha}{e}} = e^{\frac{\infty}{e}} = e^{\infty} \to e^{0} = 1$</td>
</tr>
</tbody>
</table>
1.5. The "complex to real" case $\beta<>0$ (complex $u,t$, real $s$)

Here we need not separate special ranges for $\alpha$ and/or $a$; so I display the relations between the parameters in a graph. The $x$-axis shows the sole independent parameter $\beta$; the imaginary part of $u$. From here the parameter $\alpha$ (the real part of $u$) must satisfy a functional condition dependent on $\beta$; in the graph this is the blue line. The symmetry wrt the $y$-axis shows, that the conjugate solution of $u$ applies with the same parameter $\alpha$ and gives the same result for $s$.

Functional (and uniquely dependent) on the initial choice of $\beta$ are then also the real and imaginary parts of $t = \exp(u)$. The real and imaginary parts are displayed in magenta color.

And depending on $t$ also $s = t^{1/\alpha}$ is functionally defined. It is displayed in green color, and rescaled here as $\log(\log(s))$; this scale is indicated at the right border of the graph. The minimum of the symmetric curve for $\log(\log(s))$ is

$$\log(\log(s)) = -1, \log(s) = 1/e, s = e^{1/e}.$$ 

This means, this function in $\beta$ covers all $s > e^{1/e}$ - just the region above the range of the "real-only" case.

In the following graph the red-marked points have coordinates

$$u = \alpha + \beta i = 0 + \pi i, \quad t = a + b i = 0 + i, \quad s = e^{\pi^2}.$$ 

Obviously for the two range-definitions dependend on $\beta$ (where, if $\beta=0$, $\alpha$ can freely be selected), we get the full set of real values $s>0$ for $s$.

$$\beta = 0, \quad -\infty < \alpha < +\infty \quad 0 < s < e^{1/e}$$
$$0 < \beta < \pi \quad 0 < \alpha < 3 \quad e^{1/e} \leq s < +\infty$$

Gottfried
A reference:
See also a related description for ranges of the multivalued Lambert-W-function by Corless et al. where \( \eta \) appears as the parameter \( \beta \) in my formula and \( \eta \cot \eta = \beta /\sin(\beta)\cos(\beta) \)

http://www.cs.uwaterloo.ca/research/tr/1993/03/W.pdf

The curve which separates the principal branch, \( W_0 \), from the branches \( W_1 \) and \( W_{-1} \) is

\[
\{-\eta \cot \eta + \eta i : -\pi < \eta < \pi \}
\]

(4.4)

together with \(-1\) (which is the limiting value at \( \eta = 0 \)). The curve separating \( W_1 \) and \( W_{-1} \) is simply \((-\infty, -1]\). Finally, the curves separating the remaining branches are

\[
\{-\eta \cot \eta + \eta i : 2k\pi < \pm \eta < (2k + 1)\pi \} \quad \text{for} \quad k = 1, 2, \ldots
\]

(4.5)